

# Curved spacetimes with local $\kappa$ -Poincaré dispersion relation

Leonardo Barcaroli,<sup>1,\*</sup> Lukas K. Brunkhorst,<sup>2,3,†</sup> Giulia

Gubitosi,<sup>4,‡</sup> Niccoló Loret,<sup>1,§</sup> and Christian Pfeifer<sup>5,3,2,¶</sup>

<sup>1</sup>*Dipartimento di Fisica, Università "La Sapienza" and*

*Sez. Roma1 INFN, P.le A. Moro 2, 00185 Roma, Italy*

<sup>2</sup>*Center of Applied Space Technology and Microgravity (ZARM),*

*University of Bremen, Am Fallturm, 28359 Bremen, Germany*

<sup>3</sup>*Institute for Theoretical Physics, Universität Hannover,*

*Appelstrasse 2, 30167 Hannover, Germany*

<sup>4</sup>*Theoretical Physics, Blackett Laboratory,*

*Imperial College, London SW7 2AZ, United Kingdom.*

<sup>5</sup>*Laboratory of Theoretical Physics, Institute of Physics,*

*University of Tartu, W. Ostwaldi 1, 50411 Tartu, Estonia*

We use our previously developed identification of dispersion relations with Hamilton functions on phase space to locally implement the  $\kappa$ -Poincaré dispersion relation in the momentum spaces at each point of a generic curved spacetime. We use this general construction to build the most general Hamiltonian compatible with spherical symmetry and the Planck-scale-deformed one such that in the local frame it reproduces the  $\kappa$ -Poincaré dispersion relation. Specializing to Planck-scale-deformed Schwarzschild geometry, we find that the photon sphere around a black hole becomes a thick shell since photons of different energy will orbit the black hole on circular orbits at different altitudes. We also compute the redshift of a photon between different observers at rest, finding that there is a Planck-scale correction to the usual redshift only if the observers detecting the photon have different masses.

---

\* [leonardo.barcaroli@roma1.infn.it](mailto:leonardo.barcaroli@roma1.infn.it)

† [lukas.brunkhorst@zarm.uni-bremen.de](mailto:lukas.brunkhorst@zarm.uni-bremen.de)

‡ [g.gubitosi@imperial.ac.uk](mailto:g.gubitosi@imperial.ac.uk)

§ [niccolo.loret@roma1.infn.it](mailto:niccolo.loret@roma1.infn.it)

¶ [christian.pfeifer@ut.ee](mailto:christian.pfeifer@ut.ee)

## I. INTRODUCTION

The  $\kappa$ -Poincaré algebra of symmetries, a quantum deformation of the Poincaré algebra [1–3], is one of the most intensively studied phenomenological models relevant for quantum gravity research. This is mostly because it provides a mathematically consistent example of a relativistic theory with two invariants (the speed of light and the Planck length or energy) and it produces potentially observable effects, such as an energy-dependent propagation velocity of massless particles which may be measured in the observation of  $\gamma$ -ray bursts at cosmological distances (see [4] and references therein). Geometrically, the motion of a particle admitting  $\kappa$ -Poincaré symmetry can be interpreted as happening on a flat spacetime manifold with a curved momentum space enjoying de Sitter symmetry, the Planck scale being related to the curvature of the momentum space itself [5, 6].

As already discussed in [7], in order to make the  $\kappa$ -Poincaré model more suited to describe quantum gravity effects in the cosmological framework, it is necessary to implement the  $\kappa$ -Poincaré dispersion relation on generally curved spacetimes. This entails building a model of intertwined spacetime and momentum space such that in a local frame one recovers the flat spacetime  $\kappa$ -Poincaré dispersion relation. In the local frame the  $\kappa$ -Lorentz symmetries hold, i.e. the  $\kappa$ -Poincaré symmetries except translations. By now several steps towards this goal have been achieved. The so called  $q$ -de Sitter dispersion relation implements the  $\kappa$ -Poincaré dispersion relation on de Sitter spacetime geometry [8] and is associated to a quantum deformation of the de Sitter algebra of spacetime symmetries. A first approach to a homogeneous and isotropic spacetime with  $\kappa$ -Poincaré dispersion relation was presented in [9] by gluing together slices of its de Sitter spacetime realization. Recently we could go even further. In [10] we interpreted dispersion relations as level sets of Hamilton functions on the cotangent bundle of a spacetime manifold and developed a precise notion of symmetries of dispersion relations. This enabled us to construct the most general homogeneous and isotropic dispersion relation and to identify what we called the qFLRW dispersion relation [7]. It is constructed such that in a local frame the dispersion relation reduces to the  $\kappa$ -Poincaré one, and, when the Planck-scale deformation vanishes it describes the motion of a relativistic particle on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime.

Here we show that in fact the  $\kappa$ -Poincaré dispersion relation (and the associated  $\kappa$ -Lorentz symmetries) can be realized locally on a general curved spacetime. Specifically, in section II we construct a Planck-scale-modified Hamiltonian on a general curved spacetime, such that at every point of spacetime there exists a basis of the cotangent space such that the covariantly-defined dispersion relation takes the standard  $\kappa$ -Poincaré form. This characterization is similar to the

fact that on every Lorentzian manifold there exist frames of the Lorentzian spacetime metric. In these frames the dispersion relation of point particles on curved spacetime takes the same form as on flat Minkowski spacetime. In [IIC](#) we use Hamilton equations to work out the motion in phase space of a particle with such Planck-scale-deformed dispersion relation on generic curved spacetime and in [IID](#) we derive a general formula for the redshift between two observers. In [section III](#) we specialize our model to the case of a spherically symmetric spacetime, presenting the most general Planck-scale-deformed dispersion relation compatible with these symmetries which reduces to the  $\kappa$ -Poincaré one in the local frame. It contains four free functions depending on the time and radial coordinate satisfying one algebraic constraint. Two of these functions are fixed by the undeformed metric spacetime geometry, as we show in [IIIB](#), where we deal with the Schwarzschild case. In [IIIC](#) we work out the Hamilton equations for the Schwarzschild case and finally in [IIID](#) we compute some observable effects. In particular, we show that the radius of the circular photon orbits around a black hole depend on the photon's energy and that the observed redshift of a photon moving radially in the Schwarzschild geometry is modified with respect to the standard case only if the two observers detecting the photon have different masses.

During this article we use the following notational conventions: Indices  $a, b, c, \dots$  and  $\mu, \nu, \dots$  run from 0 to 3. Latin indices denote tensor components in manifold induced coordinates, greek indices denote frame induced coordinates of the cotangent bundle. Tensorial objects on spacetime, like a spacetime metric  $g$  or a vector field  $Z$  are often interpreted as function on the cotangent bundle  $g^{-1}(p, p)$  or  $Z(p)$  which are defined as these tensors action on the 1-form  $P = p_a dx^a$ :

$$g^{-1}(p, p) = g^{ab} p_a p_b, \quad Z(p) = Z^a p_a.$$

## II. $\kappa$ -POINCARÉ MOMENTUM SPACES ON CURVED SPACETIMES

The general framework of Hamilton geometry applied to Planck-scale-modified dispersion relations is discussed in detail in our previous publications [\[7, 10\]](#). Here we only introduce the basic notation, recalling the connection between dispersion relations and level sets of Hamilton functions on the cotangent bundle of a spacetime manifold and emphasizing the link between different coordinatizations of the cotangent bundle  $T^*M$ . Subsequently, we write down the Hamilton function which implements the  $\kappa$ -Poincaré dispersion relation for free particles at every point of a generic spacetime and introduce the notion of  $\kappa$ -Lorentzian symmetry. We then briefly discuss the equations of motion induced by such Hamiltonian and we compute the redshift between any two observers.

### A. Dispersion relations as level sets of Hamilton functions

The phase space of a point particle on a curved spacetime is identified with the cotangent bundle  $T^*M$  of spacetime, consisting of all cotangent spaces  $T_q^*M$  (physically the momentum spaces) of  $M$  and  $M$  itself. An element of  $T^*M$  is a 1-form  $P$  in some cotangent space  $T_q^*M$  at  $q \in M$ . In local coordinates  $x$  around the point  $q$  we can identify the 1-form  $P$  with the following tuple

$$T_x M \ni P = p_a dx^a|_x \sim (x, p). \quad (1)$$

The labels  $(x, p)$  of points  $P$  of  $T^*M$  are so called  $x$ -manifold-induced coordinates of  $T^*M$ , which will be mainly used in this article. Observe that we could use another set of manifold induced coordinates on  $T^*M$ , namely those with respect to an arbitrary basis  $\omega^\mu$  of  $T_x^*M$ , since

$$P = p_a dx^a = \mathfrak{p}_\mu \omega^\mu = \mathfrak{p}_\mu A^\mu{}_a(x) dx^a, \quad (2)$$

where  $A$  is the transformation matrix between the bases  $dx^a$  and  $\omega^\mu$  in  $T_x^*M$ . The  $\omega$ -manifold induced coordinates of  $P \in T^*M$  are  $P \sim (x, \mathfrak{p})_\omega$ . The relation between these different coordinates on  $T^*M$  is a *local and linear* coordinate change of the manifold-induced coordinate momenta.

A Hamilton function  $H$  is a real valued function on  $T^*M$  which determines the motion of free point particles on spacetime via the Hamilton equations of motion. The level sets of  $H$  are the dispersion relations of the point particles in consideration. Moreover, as demonstrated in [10], the Hamilton function determines the geometry of phase space  $(T^*M, H)$ , i.e. the geometry of the momentum spaces and of spacetime, similarly as a metric  $g$  determines the geometry of a metric manifold  $(M, g)$ . In general this geometry is intertwined, i.e. both the geometry of spacetime and the geometry of momentum space are position and momentum dependent. In the special case of general relativity, at every point  $x \in M$ , the dispersion relation of point particles are the level sets of the Hamiltonian

$$H_g(x, p) = g^{ab}(x) p_a p_b \equiv g^{-1}(p, p). \quad (3)$$

Its special feature is that for every  $x \in M$  there exist  $\omega$ -manifold induced coordinates of  $T^*M$  such that the Hamiltonian assumes the Minkowski spacetime form

$$H_g(x, p(\mathfrak{p})) = \eta^{\mu\nu} \mathfrak{p}_\mu \mathfrak{p}_\nu. \quad (4)$$

In other words, not using the cotangent bundle terminology, there exist frames of the metric.

The geometry of  $(T^*M, H_g)$  derived from  $H_g$  yields the usual Lorentzian metric spacetime geometry on spacetime itself and a flat momentum space geometry [10].

## B. The locally $\kappa$ -Poincaré Hamiltonian

The  $\kappa$ -Poincaré dispersion relation [11] can be represented as the level sets of the Hamilton function

$$H_\kappa(x, p) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} p_t\right)^2 + e^{\ell p_t} \vec{p}^2, \quad (5)$$

where  $p_t$  and  $\vec{p}$  are, respectively, the particle's energy and spatial momentum and  $\ell = \kappa^{-1}$  is the deformation parameter, such that for  $\ell = 0$  the Hamiltonian reduces to the familiar expression of special relativity

$$H(x, p) = -p_t^2 + \vec{p}^2 = \eta^{ab} p_a p_b. \quad (6)$$

The  $\kappa$ -Poincaré Hamiltonian refers to particles on flat spacetime, so it can be considered as the Planck-scale deformation of (6). In analogy to the case of general relativity discussed at the end of the previous subsection, we will now construct a covariant Hamilton function on a generic curved spacetime that allows for  $\omega$ -manifold induced coordinates around every point  $x \in M$  such that the Hamiltonian takes the form (5).

Let  $(M, g)$  be a globally hyperbolic Lorentzian spacetime which we relabel in terms of the Hamilton function of general relativity as  $(T^*M, H_g)$  with  $H_g(x, p) = g^{ab}(x) p_a p_b \equiv g^{-1}(p, p)$ , as discussed in the previous section. Let  $Z$  be a normalized globally-defined timelike vector field on  $(M, g)$ , which can be interpreted as function  $Z(p)$  on  $T^*M$ ,

$$g(Z, Z) \equiv g_{ab}(x) Z^a(x) Z^b(x) = -1, \quad Z(p) = Z^a(x) p_a. \quad (7)$$

The  $\kappa$ -Poincaré deformation  $(T^*M, H_{Zg})$  of  $(T^*M, H_g)$  is defined by changing the Hamiltonian  $H_g$  to  $H_{Zg}$  defined by

$$H_{Zg}(x, p) \equiv -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} Z(p)\right)^2 + e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2). \quad (8)$$

We label the deformed Hamiltonian by the vector field  $Z$  since in general different choices of  $Z$  lead to different  $\kappa$ -deformed Hamiltonians. In section III, where we discuss the spherically symmetric  $\kappa$ -Poincaré phase space, we will see this freedom explicitly.

Performing a power-series expansion in  $\ell$  we find

$$H_{Zg}(x, p) = g^{-1}(p, p) + \ell Z(p) (g^{-1}(p, p) + Z(p)^2) + \mathcal{O}(\ell^2). \quad (9)$$

Thus the zeroth order of  $H_{Zg}$  is identical to the Hamilton function which determines the particle motion and the geometry of spacetime in general relativity. In  $\omega$ -manifold induced coordinates,

and  $\{\omega^\mu\}_{\mu=0}^3$  being the co-frame of  $g$  with  $\omega^0$  being dual to  $Z$ , the  $\kappa$ -deformed  $H_{Zg}$  assumes the flat  $\kappa$ -Poincaré form as desired:

$$H_{Zg}(x, \mathbf{p}) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} \mathbf{p}_0\right)^2 + e^{\ell p_t} \vec{\mathbf{p}}^2. \quad (10)$$

Employing the *local and linear* transformation of the momenta coordinates

$$\mathbf{p}_\mu = A^{-1b}{}_\mu(x) p_b \quad (11)$$

equation (10) becomes (8) again, and vice versa. The explicit calculation can be found in appendix A. This justifies to call (8) locally  $\kappa$ -Poincaré Hamiltonian or  $\kappa$ -deformed Hamiltonian.

Since locally, at every point in spacetime, we can transform the Hamiltonian (8) to its flat spacetime expression (5), we can discuss local symmetry transformations. From the flat spacetime case we know that (5) is invariant under the transformations induced by the  $\kappa$ -Poincaré algebra. However on curved spacetime the origin of the cotangent spaces is fixed, i.e. in general momenta are no longer conserved charges, thus we can no longer apply translations to the Hamiltonian. The remaining elements of the algebra, i.e. the  $\kappa$ -Poincaré boosts and rotations, constitute the  $\kappa$ -Lorentz algebra.

One further feature this Hamiltonian implements is that it immediately yields an intertwined curved geometry of spacetime and momentum space, i.e. of phase space, since its third derivatives with respect to the momenta do not vanish [10].

### C. Particle motion

Having implemented the  $\kappa$ -Poincaré dispersion relation locally on a general curved spacetime as level sets of the Hamilton function (8), we study the particle motion in phase space which is determined by the Hamilton equations of motion derived from (8). These are eight first-order ordinary differential equations which are equivalent to four second order ordinary differential equations, the Euler-Lagrange equations of the Lagrangian corresponding to the Hamiltonian in consideration. The transformation of the Hamiltonian representation of the theory to its Lagrangian counterpart is the starting point for finding a Finsler geometric formulation of the  $\kappa$ -deformed geometry of spacetime, which is investigated in several articles [12–14].

The Hamilton equations of motion of the general  $\kappa$ -deformed Hamiltonian imply the following

relation between velocities and momenta:

$$\begin{aligned}\dot{x}^a &= \bar{\partial}^a H_{Zg} \\ &= Z^a \left[ -\frac{2}{\ell} \sinh(\ell Z(p)) + \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) + 2e^{\ell Z(p)} Z(p) \right] + 2e^{\ell Z(p)} g^{ab} p_b,\end{aligned}\tag{12}$$

while the evolution of momenta is given by

$$\begin{aligned}\dot{p}_a &= -\partial_a H_{Zg} \\ &= p_q \partial_a Z^q \left[ \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) - 2e^{\ell Z(p)} Z(p) \right] - e^{\ell Z(p)} p_b p_c \partial_a g^{bc}.\end{aligned}\tag{13}$$

The latter can be written in an explicitly covariant form with respect to manifold induced coordinate transformation by introducing the Levi-Civita connection of the Lorentzian metric  $g$ :

$$\begin{aligned}\dot{p}_a &= p_q \nabla_a Z^q \left[ \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) - 2e^{\ell Z(p)} Z(p) \right] + 2e^{\ell Z(p)} p_c p^b \Gamma^c_{ba} \\ &\quad + p_q \Gamma^q_{ab} Z^b \left[ \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) - 2e^{\ell Z(p)} Z(p) \right].\end{aligned}$$

Observe that this spacetime metric is used here as an available mathematical tool to check the covariance of the equations of motion explicitly. It is not what fundamentally determines the geometry of spacetime, momentum space nor the motion of particles (in fact we can not really separate spacetime and momentum space within the phase space). The fundamental ingredient is the Hamilton function itself and when the Planck-scale corrections are introduced spacetime and momentum space are intertwined so that it is not possible to talk about a spacetime metric on its own.

Reshuffling the terms in the above equations we find

$$\begin{aligned}p_q \nabla_a Z^q \left[ \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) - 2e^{\ell Z(p)} Z(p) \right] = \\ \dot{p}_a - 2e^{\ell Z(p)} p_c p^b \Gamma^c_{ba} - p_q \Gamma^q_{ab} Z^b \left[ \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) - 2e^{\ell Z(p)} Z(p) \right].\end{aligned}\tag{14}$$

Since the Hamilton equations of motion are covariant, i.e. behave tensorial under manifold induced coordinate changes, and since the left hand side of these equations are covariant as well, the right hand side must be covariant. For  $\ell \rightarrow 0$  we obtain, as expected, the geodesic equation in its Hamilton formulation and the usual relation between momenta and velocities in general relativity

$$\dot{x}^a = 2g^{ab} p_b, \quad \dot{p}_a - 2p_c p^b \Gamma^c_{ba} = 0 \Rightarrow \ddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = \nabla_{\dot{x}} \dot{x} = 0.\tag{15}$$

We do not transform the Hamilton equations of motion of the  $\kappa$ -deformed Hamiltonian into their Euler-Lagrange form explicitly since this is a lengthy calculation not needed for the scope of this

article. If needed they can be directly calculated from the Lagrangian corresponding to the  $\kappa$ -deformed Hamiltonian. Surprisingly it is not too difficult to derive the Legendre transformation  $L(x, \dot{x}) = \dot{x}(p(x, \dot{x})) - H_{Zg}(x, p(x, \dot{x}))$  of  $H_{Zg}$  explicitly. The calculations are discussed in appendix [B](#) and yield

$$\dot{x}(p) = \frac{g(\dot{x}, \dot{x}) + g(\dot{x}, Z)^2}{\ell g(\dot{x}, Z) \pm \sqrt{2\ell^2 g(\dot{x}, Z)^2 + \ell^2 g(\dot{x}, \dot{x}) + 4}} - \frac{g(\dot{x}, Z)}{\ell} \ln \left( \frac{1}{2} (\ell g(\dot{x}, Z) \pm \sqrt{2\ell^2 g(\dot{x}, Z)^2 + \ell^2 g(\dot{x}, \dot{x}) + 4}) \right) \quad (16)$$

$$H_{Zg}(x, p(x, \dot{x})) = \frac{2}{\ell^2} - \frac{g(\dot{x}, Z)}{\ell} - \frac{4}{\ell^2} \frac{1}{(\ell g(\dot{x}, Z) \pm \sqrt{2\ell^2 g(\dot{x}, Z)^2 + \ell^2 g(\dot{x}, \dot{x}) + 4})}. \quad (17)$$

Even though these expressions are quite involved one can calculate their  $\ell \rightarrow 0$  limit and obtain

$$\dot{x}(p) = \frac{1}{2}g(\dot{x}, \dot{x}), \quad H_{Zg}(x, p(x, \dot{x})) = \frac{1}{4}g(\dot{x}, \dot{x}) \Rightarrow L(x, \dot{x}) = \frac{1}{4}g(\dot{x}, \dot{x}) \quad (18)$$

as expected.

The main qualitative difference between the Hamilton equations in general relativity and the ones for the  $\kappa$ -deformed Hamiltonian is that in general relativity the  $\dot{p}_a$  equation is only sourced by a term proportional to the Christoffel symbols, while in the  $\kappa$ -deformed case there are extra source terms. This means that, unlike in general relativity, there exists no coordinate system around every point  $q$  of spacetime such that  $\dot{p}_a = 0$  at  $q$  (i.e. it is not possible to define normal coordinates around every point). This nicely demonstrates what we already discussed in Theorem 2 of [\[10\]](#), namely that for non-homogeneous Hamiltonians a force-like term appears in the Hamilton equations dragging particles away from auto-parallel motion.

#### D. Observers and redshift

One prominent feature of physics on curved spacetimes is the gravitational redshift. Following our previous analysis done for homogeneous and isotropic models [\[7\]](#), here we investigate how the amount of redshift between two observers in a generic curved spacetime is influenced by the  $\kappa$ -deformation. In order to do so we need a notion of the frequency  $\nu_\sigma(\gamma)$  of a light ray  $\gamma$  measured by an observer  $\sigma$ .

A light ray is a solution  $\gamma(\tau) = (x_\gamma(\tau), p_\gamma(\tau))$  of the Hamilton equations of motion which satisfies  $H(\gamma) = 0$ . An observer is a curve  $\sigma(\lambda) = (x_\sigma(\lambda), p_\sigma(\lambda))$  to which a tangent vector is associated via  $\dot{x}_\sigma^a = \bar{\partial}^a H(\sigma)$  and which satisfies the following properties:

1. The energy of an observer is real for all masses and spatial momenta, i.e.  $H(\sigma) < 0$ ,



2. It is normalized, i.e.  $H(\sigma) = -m_\sigma^2 = \text{constant}$ ,

These conditions are the same conditions observers satisfy in general relativity, which can be realized in the  $\ell \rightarrow 0$  limit of the theory we are discussing. Note that we do not demand the observer's curve to be a solution of the remaining Hamilton equations, since there exist observers who are not freely falling on spacetime. However the relation between the observer's four momentum  $p_\sigma$  and the observer's tangent  $\dot{x}_\sigma$  is given via the first Hamilton equation of motion. So the observer is also subject to the  $\kappa$ -deformed dynamics, in contrast to other models considered [9, 15, 16], in which the observer is formalized just as a low-energetic (classical) worldline. In our case, however, since we are describing deformations to the particles' dynamics in a Schwarzschild-like framework later, the mass of the observer plays a crucial role, being proportional to the influence of the  $\kappa$ -deformation detected in the observer's reference frame, as we will see explicitly in [IIID 2](#).

The frequency an observer associates to the light ray is given by

$$\nu_\sigma(\gamma) = p_{\gamma a} \frac{\dot{x}_\sigma^a}{m_\sigma} = p_{\gamma a} \frac{\bar{\partial}^a H(\sigma)}{m_\sigma}. \quad (19)$$

Surely this expression only makes sense when the light ray and the observer intersect at a certain point on spacetime.

For the  $\kappa$ -Poincaré Hamiltonian  $\dot{x}$  is displayed in [\(12\)](#) so

$$\begin{aligned} \nu_\sigma(\gamma)m_\sigma = Z(p_\gamma) \left[ -\frac{2}{\ell} \sinh \left( \ell Z(p_\sigma) \right) + \ell e^{\ell Z(p_\sigma)} (g^{-1}(p_\sigma, p_\sigma) + Z(p_\sigma)^2) \right. \\ \left. + 2e^{\ell Z(p_\sigma)} Z(p_\sigma) \right] + e^{\ell Z(p_\sigma)} 2g^{-1}(p_\sigma, p_\gamma) \end{aligned} \quad (20)$$

with correct classical limit  $\ell \rightarrow 0$

$$\nu_\sigma(\gamma) = \frac{2}{m_\sigma} g^{-1}(p_\sigma, p_\gamma). \quad (21)$$

We demanded that  $H(\sigma) = -m_\sigma^2$  is constant thus we can use

$$-\frac{4}{\ell^2} \sinh \left( \frac{\ell}{2} Z(p_\sigma) \right)^2 + e^{\ell Z(p_\sigma)} (g^{-1}(p_\sigma, p_\sigma) + Z(p_\sigma)^2) = -m_\sigma^2 \quad (22)$$

to simplify the frequency to

$$\nu_\sigma(\gamma) = \frac{1}{m_\sigma} Z(p_\gamma) \left[ \frac{2}{\ell} e^{-\ell Z(p_\sigma)} - \frac{2}{\ell} - \ell m_\sigma^2 + 2e^{\ell Z(p_\sigma)} Z(p_\sigma) \right] + e^{\ell Z(p_\sigma)} \frac{2}{m_\sigma} g^{-1}(p_\sigma, p_\gamma). \quad (23)$$

This last expression can easily be used to calculate the redshift between two different observers  $\sigma_1$  and  $\sigma_2$  who intersect the light ray at different spacetime positions

$$z + 1 = \frac{\nu_{\sigma_1}(\gamma)}{\nu_{\sigma_2}(\gamma)}. \quad (24)$$

In section [IIID 2](#) we will use this formula to derive the deformation of the gravitational redshift in a  $\kappa$ -deformation of Schwarzschild geometry.

### III. SPHERICALLY SYMMETRIC $\kappa$ -DEFORMED PHASE SPACE

In our previous article [10] we gave a detailed account of the notion of symmetry in Hamilton geometry. Summarizing, a Hamiltonian  $H(x, p)$  is invariant under the action of certain diffeomorphisms  $\Phi$  on phase space if the vector field  $X_\Phi$  which induces this diffeomorphism annihilates the Hamiltonian

$$X_\Phi(H) = 0. \quad (25)$$

Particularly interesting are those diffeomorphisms of phase space which are induced by a diffeomorphism of the spacetime manifold. In this case the symmetry condition becomes

$$X^C(H) \equiv (\xi^a \partial_a - p_q \partial_a \xi^q \bar{\partial}^a) H = 0, \quad (26)$$

where  $X = \xi^a(x) \partial_a$  is the vector field which induces the diffeomorphism of spacetime. The details of the derivation of this symmetry condition can be found in [10], while an application in the context of homogeneous and isotropic geometries is discussed in [7]. In the following we use this construction to define general spherically symmetric Hamiltonians.

#### A. The general case

In order to study spherically symmetric phase spaces it is most convenient to use spherical coordinates  $(t, r, \theta, \phi, p_t, p_r, p_\theta, p_\phi)$ . The generators of rotations of spacetime are

$$X_1 = \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \quad (27)$$

$$X_2 = -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi \quad (28)$$

$$X_3 = \partial_\phi. \quad (29)$$

Their complete lifts are displayed in the appendix C. Evaluating equation (26) we find, with the same techniques already used in the homogeneous and isotropic case [7], that the most general spherically symmetric Hamiltonian must take the form

$$H(x, p) = H(t, p_t, r, p_r, w(\theta, p_\theta, p_\phi)) \text{ with } w^2 = p_\theta^2 + \frac{1}{\sin^2 \theta} p_\phi^2. \quad (30)$$

As one could expect, the form of the Hamiltonian is less constrained compared to the homogeneous and isotropic case [7]. This freedom translates in the appearance of several free functions in the

most general third-order polynomial expansion around the standard metric dispersion relation:

$$H(x, p) = -A(t, r)p_t^2 + B(t, r)p_r^2 + \frac{1}{r^2}w^2 + \ell \left( C(t, r)p_t^3 + D(t, r)p_t^2 p_r + E(t, r)p_t p_r^2 + F(t, r)p_r^3 + G(t, r)p_t w^2 + H(t, r)p_r w^2 \right) + \mathcal{O}(\ell^2). \quad (31)$$

Since we are interested in building a Hamiltonian that reduces to the  $\kappa$ -Poincaré one in the local frame we will have a reduced freedom compared to this general case. In particular, we want to construct a Hamiltonian that, besides having spherical symmetry, can be written in the form (8). The general  $\kappa$ -deformed Hamiltonian (8) is built out of two elements: a spacetime metric term  $g^{-1}(p, p)$  and a vector field term  $Z(p)$ . The most general spherically-symmetric metric for the base manifold can be written, after an appropriate choice of coordinates, as

$$g^{-1}(p, p) = -a(t, r)p_t^2 + b(t, r)p_r^2 + \frac{1}{r^2}w^2. \quad (32)$$

On the other hand, in order to respect spherical symmetry, the vector field must take the form

$$Z(p) = c(t, r)p_t + d(t, r)p_r, \quad (33)$$

subject to the condition  $g(Z, Z) = -1$ , which yields

$$-\frac{c(t, r)^2}{a(t, r)} + \frac{d(t, r)^2}{b(t, r)} = -1. \quad (34)$$

Plugging these objects into the  $\kappa$ -deformed Hamiltonian (8) results in the most general spherically symmetric  $\kappa$ -deformed Hamiltonian:

$$H_{Zg} = \frac{4}{\ell^2} \sinh \left( \frac{\ell}{2} (cp_t + dp_r) \right)^2 + e^{\ell(cp_t + dp_r)} ((-a + c^2)p_t^2 + 2cdp_r p_t + (b + d^2)p_r^2 + \frac{1}{r^2}w^2), \quad (35)$$

where we suppressed the arguments of the functions  $a, b, c, d$  for the sake of readability.

The functions  $c$  and  $d$ , intertwined by (34), identify a family of  $\kappa$ -deformations of the phase space of a spherically symmetric spacetime. One could hope that some fundamental mechanism derived from a complete theory of quantum gravity would single out one specific correct form of the deformation.

One the other hand, if one restricts to specific spherically-symmetric spacetimes, it is not always the case that there exists such freedom in the definition of the  $\kappa$ -deformation. For example, including further symmetries like in the homogeneous and isotropic case discussed in [7], the only normalized homogeneous and isotropic vector field evaluated on a 1-form  $P = p_a dx^a$  is

$$Z(p) = p_t. \quad (36)$$

Then the unique homogeneous and isotropic  $\kappa$ -deformed Hamiltonian was found to be

$$H_{qFLRW} = -\frac{4}{\ell} \sinh\left(\frac{\ell}{2} p_t\right)^2 + e^{\ell p_t} a(t)^{-2} \left( (1 - kr^2) p_r^2 + \frac{1}{r^2} w^2 \right). \quad (37)$$

Here no additional degrees of freedom in addition to the scale factor of the  $FLRW$  metric, which is determined by the Einstein equations, appear.

In the following we specialize to the  $\kappa$ -deformation of the most famous spherically symmetric solution of Einstein's equations, the Schwarzschild geometry.

### B. The $\kappa$ -deformation of Schwarzschild geometry

In the Schwarzschild solution of general relativity the functions which determine the spacetime metric are

$$a(t, r) = \frac{1}{1 - \frac{r_s}{r}}, \quad b(t, r) = a(t, r)^{-1} = 1 - \frac{r_s}{r}, \quad (38)$$

where  $r_s$  is the Schwarzschild radius. Thus the functions  $c$  and  $d$  appearing in the timelike vector field  $Z$  which defines the deformation of the classical phase space, eq. (33), must satisfy

$$-\left(1 - \frac{r_s}{r}\right) c(t, r)^2 + \frac{d(t, r)^2}{\left(1 - \frac{r_s}{r}\right)} = -1, \quad (39)$$

according to equation (34). Following the discussion of the previous section we can write down the general spherically symmetric  $\kappa$ -deformation of the phase space of Schwarzschild spacetime

$$H_{ZSchw}(x, p) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} (cp_t + dp_r)\right)^2 + e^{\ell(cp_t + dp_r)} \left[ \left( -\frac{1}{1 - \frac{r_s}{r}} + c^2 \right) p_t^2 + 2cdp_r p_t + \left( 1 - \frac{r_s}{r} + d^2 \right) p_r^2 + \frac{1}{r^2} w^2 \right]. \quad (40)$$

In the rest of this section we omit the subscript  $ZSchw$  for the sake of readability. As already mentioned we find a family of deformations defined by the function  $c$  and  $d$  subject to the condition (39). This result demonstrates the importance of our general construction in section IIB, since without the insight that a vector field parametrizes the possible  $\kappa$ -Poincaré deformations we may not have found this general class of  $\kappa$ -deformations of Schwarzschild geometry.

### C. Motion in phase space

To study observable consequence of the  $\kappa$ -deformation of Schwarzschild geometry we now discuss the equations of motion for point particles.

In general relativity the Einstein equations guarantee that every spherically symmetric solution of the equations is static, also known as Birkhoff's theorem. Since so far we have not developed further the dynamics which the  $\kappa$ -deformation of a classical spacetime geometry has to satisfy, in the following we assume for simplicity that  $c$  and  $d$  do not depend on  $t$ , i.e. that  $\partial_t$  induces yet another symmetry of  $H$ .

Due to the symmetry of the geometry which we are studying there exist several constants of motion, one for each generator of symmetry  $X_I$ , displayed in equations (27) to (29), to which we add the generator of time translations  $\partial_t$ . The constants of motion are found as  $X_I(P) = X_I^a(x)p_a$ . In fact, it is easy to see that this object is constant along the solutions to the Hamilton equations of motion. One then finds the constants of motion:

$$E = p_t, \quad L = p_\phi, \quad K_1 = \sin \phi p_\theta + \cot \theta \cos \phi p_\phi, \quad K_2 = -\cos \phi p_\theta + \cot \theta \sin \phi p_\phi. \quad (41)$$

We can use these constants to restrict the motion of particles to the equatorial plane, fixing  $\theta = \frac{\pi}{2}$  and  $p_\theta = 0$ . For this case  $L = p_\phi = w$ . Moreover  $H$  itself is another constant of motion representing the dispersion relation

$$\begin{aligned} -m^2 = & -\frac{4}{\ell^2} \sinh \left( \frac{\ell}{2} (cp_t + dp_r) \right)^2 \\ & + e^{\ell(cp_t + dp_r)} \left[ \left( -\frac{1}{1 - \frac{r_s}{r}} + c^2 \right) p_t^2 + 2cdp_r p_t + \left( 1 - \frac{r_s}{r} + d^2 \right) p_r^2 + \frac{1}{r^2} w^2 \right]. \end{aligned} \quad (42)$$

Under these conditions the non-trivial Hamilton equations of motion are

$$\dot{t} = \bar{\partial}^t H, \quad \dot{p}_r = -\partial_r H, \quad \dot{r} = \bar{\partial}^r H, \quad \dot{\phi} = \bar{\partial}^\phi H. \quad (43)$$

Solving analytically the equations of motion is not possible, so, in order to get a first impression of the sort of effects caused by  $\kappa$ -deformations of Schwarzschild spacetime geometry we choose  $c = \frac{1}{\sqrt{|1 - \frac{r_s}{r}|}}$  in the region  $r > r_s$ , i.e. outside the classical horizon, for which equation (39) implies  $d = 0$ . A thorough analysis of the implications of general  $\kappa$ -deformations of Schwarzschild geometry, parametrized by the functions  $c$  and  $d$ , will be discussed in an upcoming separate article.

#### D. Observable effects in $d = 0$ $\kappa$ -deformed Schwarzschild geometry

Choosing  $c = \frac{1}{\sqrt{|1 - \frac{r_s}{r}|}} \equiv \frac{1}{\sqrt{A}}$ ,  $r > r_s$  and thus  $d = 0$ , the  $\kappa$ -deformed Schwarzschild Hamiltonian takes the form:

$$H(x, p) = -\frac{4}{\ell^2} \sinh \left( \frac{\ell}{2} \frac{p_t}{\sqrt{A}} \right)^2 + e^{\frac{\ell p_t}{\sqrt{A}}} \left( A p_r^2 + \frac{1}{r^2} w^2 \right). \quad (44)$$

Using this specific choice of the free functions allows to study some relevant features of the model explicitly. In the following we focus on the effects of the deformation on the circular orbits around the origin with radius larger than  $r_s$ , and on the redshift between stationary observers.

### 1. Circular particle motion

The relevant Hamilton equations in the study of circular motion are the ones associated to the radial coordinate and momentum. Moreover, the on-shell condition  $H = -m^2$  relates the particle's energy  $p_t$  to the radial and angular momenta:

$$\begin{aligned} \frac{p_t}{\sqrt{A}} &= -\frac{1}{\ell} \ln \left( 1 + \frac{\ell^2 m^2}{2} \pm \ell \sqrt{m^2 \left( \frac{\ell^2 m^2}{4} + 1 \right) + \frac{w^2}{r^2} + p_r^2 A} \right) \\ &\rightarrow -\frac{1}{\ell} \ln \left( 1 + \frac{\ell^2 m^2}{2} + \ell \sqrt{m^2 \left( \frac{\ell^2 m^2}{4} + 1 \right) + \frac{w^2}{r^2} + p_r^2 A} \right), \end{aligned} \quad (45)$$

where the sign was chosen so to have  $(\frac{p_t}{\sqrt{A}})^2 = -m^2$  for observers with  $p_r = w = 0$  in the  $\ell = 0$  limit.

A stable circular orbit is characterized by constant radial coordinate,  $\dot{r} = 0$ . Then from the Hamilton equation for  $\dot{r}$  it follows that the radial momentum must be constantly vanishing:

$$0 = \dot{r} = \bar{\partial}^r H = 2A p_r e^{\frac{\ell p_t}{\sqrt{A}}} \Rightarrow p_r = 0. \quad (46)$$

This of course also implies that  $\dot{p}_r = 0$ . Using the Hamilton equation for the radial momentum:

$$0 = \dot{p}_r = -\partial_r H = -\frac{1}{\ell} \sinh \left( \frac{\ell p_t}{\sqrt{A}} \right) \frac{r_s}{r^2} \frac{p_t}{A^{\frac{3}{2}}} - e^{\frac{\ell p_t}{\sqrt{A}}} \left( \frac{r_s}{r^2} p_r^2 - \frac{2w^2}{r^3} \right) + \frac{\ell}{2} \frac{p_t}{A^{\frac{3}{2}}} \frac{r_s}{r^2} e^{\frac{\ell p_t}{\sqrt{A}}} \left( A p_r^2 + \frac{w^2}{r^2} \right) \quad (47)$$

$$= -\frac{1}{\ell} \sinh \left( \frac{\ell p_t}{\sqrt{A}} \right) \frac{r_s}{r^2} \frac{p_t}{A^{\frac{3}{2}}} + e^{\frac{\ell p_t}{\sqrt{A}}} \frac{2w^2}{r^3} + \frac{\ell}{2} \frac{p_t}{A^{\frac{3}{2}}} \frac{r_s}{r^4} w^2 e^{\frac{\ell p_t}{\sqrt{A}}}, \quad (48)$$

where in the second line we used  $p_r = 0$ . Before solving for  $r$ , we can simplify this expression further by using the mass-shell constraint (45) to remove the  $p_t$  dependence:

$$\frac{\frac{1}{\ell} \ln \left( 1 + \frac{\ell^2 m^2}{2} + \ell P_m \right)}{A} 2r_s r P_m \left( 1 + \frac{\ell^2 m^2}{2} + \ell P_m \right) - 4w^2 = 0, \quad (49)$$

where we multiplied everything by  $2r^3 e^{-\frac{\ell p_t}{\sqrt{A}}}$  and we defined  $P_m = \sqrt{m^2 \left( \frac{\ell^2 m^2}{4} + 1 \right) + \frac{w^2}{r^2}}$ . In the massless limit this becomes:

$$\frac{\frac{1}{\ell} \ln \left( 1 + \ell \frac{w}{r} \right)}{A} 2r_s w \left( 1 + \ell \frac{w}{r} \right) - 4w^2 = 0, \quad (50)$$

In general, the equations (49) and (50) are not solvable analytically, so we continue our study perturbatively. The above equations read, up to first order in  $\ell$ :

$$\frac{r_s}{r - r_s} \left( 2m^2 r^2 + 2w^2 \left( 3 - 2\frac{r}{r_s} \right) + \ell (w^2 + 2m^2 r^2) \sqrt{m^2 + \frac{w^2}{r^2}} \right) = 0, \quad (51)$$

for the massive case, and

$$w^2 \left( -4 + 2\frac{r_s}{r - r_s} + \ell w \frac{r_s}{r(r - r_s)} \right) = 0 \quad (52)$$

for the massless case. Solving for  $r$  one finds the radius of circular orbits for massive particles:

$$r_m = \frac{w^2}{m^2 r_s} \left( 1 - \sqrt{1 - 3 \left( \frac{r_s m}{w} \right)^2} \right) + \frac{\ell}{4} w^2 m \sqrt{1 + \left( \frac{w}{m r_m^0} \right)^2} \frac{(4r_m^0 - 5r_s)}{(w^2 - r_m^0 r_s m^2)}, \quad (53)$$

where  $r_m^0 = \lim_{\ell \rightarrow 0} r_m$ . In the massless limit this becomes:

$$r_{m=0} = \frac{3}{2} r_s + \ell \frac{w}{6}. \quad (54)$$

This last results indicates that the photon sphere, which is universal in Schwarzschild geometry, is in fact dependent on the angular momentum of the photons once the Planck-scale deformation is introduced, so that photons with different energy are allowed to orbit a black hole at different altitudes.

## 2. Redshift

Our goal here is to compute the change in the energy of a photon as measured by two different observers,  $\sigma_1$  and  $\sigma_2$ , at rest. The observers are characterized by their spacetime coordinates and momenta:  $\sigma_i = (x_{\sigma_i}, p_{\sigma_i})$ ,  $i = 1, 2$ . Since the observers are at rest only the time component of their four-momentum is nonzero:  $p_{\sigma_i} = (p_{\sigma_i t}, 0, 0, 0)$ . In this case the mass-shell constraint given by the Hamiltonian reads:

$$H(x_{\sigma_i}, p_{\sigma_i}) = -\frac{4}{\ell^2} \sinh \left( \frac{\ell}{2} \frac{p_{\sigma_i t}}{\sqrt{A(x_{\sigma_i})}} \right)^2 = -m_{\sigma_i}^2. \quad (55)$$

This constraint implies that the four-momentum of the observers is related to their position and mass via  $p_{\sigma_i t} = \sqrt{A(x_{\sigma_i})} Q_{\sigma_i}$ , with  $Q_{\sigma_i} \equiv -\frac{1}{\ell} \ln \left( 1 + \frac{\ell^2 m_{\sigma_i}^2}{2} + \ell m_{\sigma_i} \sqrt{1 + \frac{\ell^2 m_{\sigma_i}^2}{4}} \right)$  being a constant.

Having defined the observers, we can use equation (23) to obtain the frequencies that each of them associates to the photon:

$$\begin{aligned} \nu_{\sigma_i}(\gamma) &= \frac{1}{m_{\sigma_i}} \frac{p_{\gamma t | \sigma_i}}{\sqrt{A(x_{\sigma_i})}} \left[ \frac{2}{\ell} e^{-\ell Q_{\sigma_i}} - \frac{2}{\ell} - \ell m_{\sigma_i}^2 \right] \\ &= \frac{p_{\gamma t | \sigma_i}}{\sqrt{A(x_{\sigma_i})}} \left[ 2 \sqrt{1 + \frac{\ell^2 m_{\sigma_i}^2}{4}} \right] \end{aligned} \quad (56)$$

The time component of the momentum of the photon at the position of the observer  $\sigma_i$  is given by  $p_{\gamma t}|_{\sigma_i}$ . Since the light trajectory  $\gamma$  is a solution of the Hamilton equations of motion,  $p_{\gamma t}$  is constant along  $\gamma$ . In particular,  $p_{\gamma t}$  has the same value at the intersection point with  $\sigma_1$  and at the intersection point with  $\sigma_2$ , so  $p_{\gamma t}|_{\sigma_1} = p_{\gamma t}|_{\sigma_2} = p_{\gamma t}$ . The redshift of the photon between the two observers is thus given by

$$z + 1 = \frac{\nu_{\sigma_1}(\gamma)}{\nu_{\sigma_2}(\gamma)} = \frac{\sqrt{A_2}}{\sqrt{A_1}} \frac{\sqrt{1 + \frac{\ell^2 m_{\sigma_1}^2}{4}}}{\sqrt{1 + \frac{\ell^2 m_{\sigma_2}^2}{4}}} = \sqrt{\frac{1 - \frac{r_s}{r_2}}{1 - \frac{r_s}{r_1}}} \frac{\sqrt{1 + \frac{\ell^2 m_{\sigma_1}^2}{4}}}{\sqrt{1 + \frac{\ell^2 m_{\sigma_2}^2}{4}}} \quad (57)$$

$$\simeq \sqrt{\frac{1 - \frac{r_s}{r_2}}{1 - \frac{r_s}{r_1}}} \left( 1 + \frac{\ell^2}{8} (m_{\sigma_1} - m_{\sigma_2})(m_{\sigma_1} + m_{\sigma_2}) \right), \quad (58)$$

where in the last step we only kept the lowest order  $\ell$ -correction. Thus for two static observers the redshift of a photon is identical to the one in Schwarzschild geometry to all orders in  $\ell$ , if the observers have the same mass. Otherwise, if the observers have different masses, then they measure a redshift which departs from the standard result to second order in  $\ell$ . This influence of the mass of the observers on the redshift is due to the fact that we assumed that the observers are also subject to the  $\kappa$ -deformed dynamics. If one were to assume that observers follow the dynamics of the general relativistic Hamiltonian (3), or that the observers have negligible masses, then there would be again no additional effect compared to the usual redshift in Schwarzschild geometry.

Surely the results of this section highly depend on the specific choice of observers and of the vector field  $Z$  (remember that the possible deformations of Schwarzschild geometries encoded by the vector field  $Z$  depend on two free functions of spacetime coordinates, which we fixed at the beginning of this subsection III D in order to have a workable example). In general we would expect that the Planck-scale deformation would alter the gravitational redshift of photons in spherical symmetry also for equal-mass observers, as it is the case in the homogeneous and isotropic cosmological situation discussed in [7].

#### IV. DISCUSSION

We used the insights we gained in the local implementation of the  $\kappa$ -Poincaré dispersion relation on homogeneous and isotropic spacetimes [7] to extend our findings to general curved spacetimes. The key result of our work is the construction of a phase space in which locally one can identify a spacetime with  $\kappa$ -Lorentz symmetry, i.e.  $\kappa$ -Poincaré symmetries excluding translations. The implementation of this local symmetry via the level sets of a Hamilton function on the point particle phase space causes the geometry of spacetime and the geometry of momentum space to be



intertwined into a geometry of phase space.

In equation (8) we presented the locally  $\kappa$ -Poincaré Hamilton function which deserves its name by the fact that at every point on spacetime there exists a local basis of the cotangent spaces of the spacetime manifold such that the level sets of the Hamilton function assume the form of the  $\kappa$ -Poincaré dispersion relation. This is the direct generalization of local Lorentz invariance of the geometry of spacetime to local  $\kappa$ -Lorentz invariance. This explicit construction of the  $\kappa$ -Poincaré Hamilton function will allow us to study the mathematical differential geometric structure of the phase space geometry in the future. In particular, the local frame bundle properties of spacetime are of interest since equivalent frames are no longer identified with linear transformations like Lorentz transformations but with the partly non-linear  $\kappa$ -Lorentz transformations, the  $\kappa$ -Poincaré boosts and rotations.

Having established the notion of a general  $\kappa$ -deformed phase space we studied the motion of test particles on such a background. The modification of the geodesic equation was presented in equation (13). As already stated when we introduced Hamiltonian geometry in [10], there appears a force-like term in the equations of motion which can not be absorbed into the geometry of spacetime. Thus there exists no local coordinate system such that the equations of motion locally reduce to  $\ddot{x} + \mathcal{O}(x^2) = 0$  as they do in normal coordinates in the undeformed spacetime geometry. Also generalizations of normal coordinates, as they were discussed in the context of Finsler geometry in [17] and [18], do not exist. To complete the discussion on particle motion on the  $\kappa$ -deformed phase space geometry we derived the Lagrangian formulation of point particle motion. This can be used as starting point for the derivation of a Finslerian version of the locally  $\kappa$ -deformed spacetime geometry in the future, as it was done for particular  $\kappa$ -deformed geometries in [12–14].

In the second half of this article we derived the most general form of the locally  $\kappa$ -Poincaré Hamilton function compatible with spherical symmetry. We obtained a Hamilton function defined in terms of four free functions of the time and radial coordinate, two of which are fixed by the specific spacetime geometry on which the deformation is based. The presence of the other two free functions is due to the fact that the timelike vector field which is necessary to define the Hamilton function is not fixed by the available symmetry constraints. This is to be contrasted with the homogeneous and isotropic case [7], where the symmetry constraints were sufficient to fully determine the form of the deformation.

We studied observable predictions of the model in the special case of deformations of the Schwarzschild geometry, where the vector field defining the deformation was chosen as the tan-

gent of the standard observer at rest in Schwarzschild geometry. In an upcoming article we will investigate the influence of the choice of this vector field on observables in more detail. The freedom in the choice of the vector field defining the deformed Hamiltonian may be related to the deformed boosts which underly the  $\kappa$ -deformed spacetime geometry, in the sense that the deformed boost may map one choice of  $Z$  to another. This will be matter of investigation in future work. For our choice of  $\kappa$ -deformed Schwarzschild geometry we studied two possibly observable features: the radius of photon orbits around the spherical symmetric black hole (known as photon sphere in the standard case) and the gravitational redshift between two observers at rest with respect to each other and with respect to the black hole horizon. For the first observable we found that the photon sphere, which is universal for all photons in Schwarzschild geometry, becomes momentum dependent. In particular, photons with a different angular momentum have circular orbits at different altitudes (54). For the redshift we found that corrections to the standard Schwarzschild case emerge only at the second order in the deformation parameter, (57). Moreover, these corrections are proportional to the difference of the masses of the observers measuring the frequency of the photon and they only exist if one assumes that the observers enjoy the same deformed symmetries as the photon itself.

In an upcoming work we will study the spherically symmetric  $\kappa$ -Poincaré deformed spacetime geometry in further detail to derive observable implications in solar system and black hole observations, like perihelion shifts, light deflections, the horizon and the singularity. Further interesting studies which are now in reach are locally  $\kappa$ -deformed spacetime geometries with any desired symmetry, like axial symmetry, as generalization of the spherically symmetric case.

Besides these phenomenological studies, one can further develop our method to locally implement more general dispersion relations on curved spacetime, generalizing the  $\kappa$ -Poincaré case that was studied here. The procedure to be applied would be to identify four basis vector fields  $\{Z_i\}_{i=0}^3$  on spacetime which represent, when applied to a four momentum  $Z_i(p)$ , the different Cartesian momentum components  $p_i = Z_i(p)$ . This sort of generalization would be particularly interesting since it would allow to compare predictions concerning black hole physics obtained in the framework of Hamilton geometry to the ones obtained using rainbow gravity as a formalization of Planck-scale effects [19–22].

## ACKNOWLEDGMENTS

CP gratefully thanks the Center of Applied Space Technology and Microgravity (ZARM) at the University of Bremen for their kind hospitality and acknowledges partial support of the European Regional Development Fund through the Center of Excellence TK133 “The Dark Side of the Universe”. GG acknowledges support from the John Templeton Foundation. LKB acknowledges the support by a Ph.D. grant of the German Research Foundation within its Research Training Group 1620 *Models of Gravity*. NL acknowledges partial support from the 000008 15 RS *Avvio alla ricerca* 2015 fellowship (by the Italian ministry of university and research).

## Appendix A: Transforming the $\kappa$ -Poincaré Hamiltonian to its frame representation

In section II B we introduced the general form of the  $\kappa$ -Poincaré dispersion relation as deformation of any given Lorentzian metric spacetime geometry. Here we display the calculation which connects the general covariant form of the  $\kappa$ -Poincaré Hamiltonian (8) with its frame counterpart (10).

A basis change on  $T_x^*M$  yields

$$P = p_a dx^a = \mathbf{p}_\mu \omega^\mu = \mathbf{p}_\mu A^\mu_a(x) dx^a \Leftrightarrow \mathbf{p}_\mu = A^{-1a}_\mu(x) p_a, \quad (\text{A1})$$

where the  $\omega$  basis shall be a co frame of the metric  $g$ , i.e.  $g = \eta_{\mu\nu} \omega^\mu \otimes \omega^\nu$ .

The  $\kappa$ -Poincaré Hamiltonian in frame coordinates on  $T^*M$  shall be

$$H_{Zg}(x, p) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} \mathbf{p}_0\right)^2 + e^{\ell \mathbf{p}_0} \vec{\mathbf{p}}^2, \quad (\text{A2})$$

which can be rewritten according to the transformation defined above ( $\alpha, \beta = 1, 2, 3$ )

$$H_{Zg}(x, p) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} p_a A^{-1a}_0\right)^2 + e^{\ell p_a A^{-1a}_0} (p_a A^{-1a}_\alpha) (p_a A^{-1a}_\beta) \delta^{\alpha\beta}. \quad (\text{A3})$$

Adding the zero  $0 = e^{\ell p_a A^{-1a}_0} ((p_a A^{-1a}_0)(p_b A^{-1b}_0) - (p_a A^{-1a}_0)(p_b A^{-1b}_0))$  and expanding the metric

$$\begin{aligned} g^{-1}(p, p) &= g^{ab} p_a p_b = g^{ab} A^\mu_a A^\nu_b \mathbf{p}_\mu \mathbf{p}_\nu = \eta^{\mu\nu} \mathbf{p}_\mu \mathbf{p}_\nu \\ &= \eta^{\mu\nu} (p_a A^{-1a}_\mu) (p_b A^{-1b}_\nu) = \delta^{\alpha\beta} (p_a A^{-1a}_\alpha) (p_b A^{-1b}_\beta) - (p_a A^{-1a}_0) (p_b A^{-1b}_0) \end{aligned} \quad (\text{A4})$$

allows us to write

$$H_{Zg}(x, p) = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} p_a A^{-1a}_0\right)^2 + e^{\ell p_a A^{-1a}_0} \left( g^{-1}(p, p) + (p_a A^{-1a}_0) (p_b A^{-1b}_0) \right). \quad (\text{A5})$$

Finally introduce  $Z = A^{-1a}_0 \partial_a$  to obtain the desired expression

$$H(x, p)_{Zg} = -\frac{4}{\ell^2} \sinh\left(\frac{\ell}{2} Z(p)\right)^2 + e^{\ell Z(p)} \left( g^{-1}(p, p) + Z(p)^2 \right), \quad (\text{A6})$$

with  $Z(p) = Z^a p_a$  and  $g^{-1}(p, p) = g^{ab} p_a p_b$ .

## Appendix B: The $\kappa$ -Poincaré Lagrangian

In section II C we discussed the Hamilton equations of motion of the general  $\kappa$ -Poincaré Hamiltonian. Here we demonstrate how the corresponding Lagrangian can be obtained from which

one can derive the second order Euler-Lagrange equations. The Legendre transformation from the Hamiltonian to the Lagrangian involves the terms

$$L(x, \dot{x}) = \dot{x}(p(x, \dot{x})) - H(x, p(x, \dot{x})) \quad (\text{B1})$$

which we will derive now.

In (12) we already found

$$\begin{aligned} \dot{x}^a &= \bar{\partial}^a H \\ &= Z^a \left[ -\frac{2}{\ell} \sinh(\ell Z(p)) + \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) + 2e^{\ell Z(p)} Z(p) \right] + e^{\ell Z(p)} 2g^{ab} p_b. \end{aligned} \quad (\text{B2})$$

Contracting this equation with  $Z$  yields

$$g(\dot{x}, Z) = \frac{2}{\ell} \sinh(\ell Z(p)) - \ell e^{\ell Z(p)} (g^{-1}(p, p) + Z(p)^2) \quad (\text{B3})$$

which allows us to write

$$\dot{x}^a = Z^a \left[ -g(\dot{x}, Z) + 2e^{\ell Z(p)} Z(p) \right] + e^{\ell Z(p)} 2g^{ab} p_b, \quad (\text{B4})$$

and

$$\dot{x}(p) = -Z(p)g(\dot{x}, Z) + 2e^{\ell Z(p)} (Z(p)^2 + g^{-1}(p, p)). \quad (\text{B5})$$

as well as

$$g(\dot{x}, \dot{x}) = -g(\dot{x}, Z)^2 + 2e^{\ell Z(p)} (Z(p)g(\dot{x}, Z) + \dot{x}(p)) \quad (\text{B6})$$

$$= -g(\dot{x}, Z)^2 + 4e^{2\ell Z(p)} (Z(p)^2 + g^{-1}(p, p)) \quad (\text{B7})$$

$$= -g(\dot{x}, Z)^2 + 4e^{\ell Z(p)} \left( \frac{2}{\ell^2} \sinh(\ell Z(p)) - \frac{g(\dot{x}, Z)}{\ell} \right) \quad (\text{B8})$$

$$= -g(\dot{x}, Z)^2 - \frac{4}{\ell} e^{\ell Z(p)} g(\dot{x}, Z) + \frac{4}{\ell^2} (e^{2\ell Z(p)} - 1) \quad (\text{B9})$$

The last equation can be reformulated as quadratic equation for  $e^{\ell Z(p)}$

$$0 = e^{2\ell Z(p)} - \ell e^{\ell Z(p)} g(\dot{x}, Z) - \frac{\ell^2}{4} (g(\dot{x}, \dot{x}) + g(\dot{x}, Z)^2) - 1. \quad (\text{B10})$$

with solution

$$e^{\ell Z(p)} = \frac{\ell}{2} g(\dot{x}, Z) \pm \sqrt{\frac{\ell^2}{2} g(\dot{x}, Z)^2 + \frac{\ell^2}{4} g(\dot{x}, \dot{x}) + 1} \quad (\text{B11})$$

$$= \frac{1}{2} (\ell g(\dot{x}, Z) \pm \sqrt{2\ell^2 g(\dot{x}, Z)^2 + \ell^2 g(\dot{x}, \dot{x}) + 4}) \quad (\text{B12})$$

$$Z(p) = \frac{1}{\ell} \ln \left( \frac{1}{2} (\ell g(\dot{x}, Z) \pm \sqrt{2\ell^2 g(\dot{x}, Z)^2 + \ell^2 g(\dot{x}, \dot{x}) + 4}) \right). \quad (\text{B13})$$

Finally we can use the terms we found to solve (B4) for  $p(x, \dot{x})$

$$p_a(x, \dot{x}) = \frac{1}{2}g_{ab}\dot{x}^b e^{-\ell Z(p)} - \frac{1}{2}g_{ab}Z^b \left[ -e^{-\ell Z(p)}g(\dot{x}, Z) + 2Z(p) \right] \quad (\text{B14})$$

$$= \frac{1}{2}e^{-\ell Z(p)} \left( g_{ab}\dot{x}^b + g_{ab}Z^b g(\dot{x}, Z) \right) - g_{ab}Z^b Z(p) \quad (\text{B15})$$

$$= \frac{g_{ab}\dot{x}^b + g_{ab}Z^b g(\dot{x}, Z)}{\ell g(\dot{x}, Z) \pm \sqrt{2\ell g(\dot{x}, Z)^2 + \ell g(\dot{x}, \dot{x}) + 4}} \quad (\text{B16})$$

$$- \frac{g_{ab}Z^b}{\ell} \ln \left( \frac{1}{2}(\ell g(\dot{x}, Z) \pm \sqrt{2\ell g(\dot{x}, Z)^2 + \ell g(\dot{x}, \dot{x}) + 4}) \right). \quad (\text{B17})$$

Contracting this expression with  $\dot{x}^a$  yields the desired equation (16). Equation (17) is obtained by solving (B3) for

$$e^{\ell Z(p)}(g^{-1}(p, p) + Z(p)^2) = \frac{2}{\ell^2} \sinh \left( \ell Z(p) \right) - \frac{g(\dot{x}, Z)}{\ell}, \quad (\text{B18})$$

plugging this result into the Hamiltonian (8) and inserting (B12) afterwards.

### Appendix C: The lifts of the symmetry generating vector fields to phase space

In section III A we used the lifts of the vector fields which generate spherical symmetry on spacetime to derive the most general spherically symmetric Hamilton function on phase space. These lifts

$$X_I^C = \xi^a \partial_a - p_q \partial_a \xi^q \bar{\partial}^a \quad (\text{C1})$$

of the vector fields  $X_I = \xi_I^a(x) \partial_a$ ,  $I = 1, 2, 3$  (see equations (27) to (29)) are given by

$$\begin{aligned} X_1^C &= \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \\ &+ \frac{\cos \phi}{\sin \theta^2} p_\phi \bar{\partial}^\theta - \left( \cos \phi p_\theta - \cot \theta \sin \phi p_\phi \right) \bar{\partial}^\phi, \end{aligned} \quad (\text{C2})$$

$$\begin{aligned} X_2^C &= -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi \\ &+ \frac{\sin \phi}{\sin \theta^2} p_\phi \bar{\partial}^\theta - \left( \sin \phi p_\theta + \cot \theta \cos \phi p_\phi \right) \bar{\partial}^\phi, \end{aligned} \quad (\text{C3})$$

$$X_3^C = \partial_\phi. \quad (\text{C4})$$

It can be easily checked by direct calculation that the Hamiltonian (30) satisfies  $X_I^C(H) = 0$  for all  $I = 1, 2, 3$ .

- 
- [1] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoy, “Q deformation of Poincare algebra,” *Phys.Lett.* **B264** (1991) 331–338.
  - [2] J. Lukierski, A. Nowicki, and H. Ruegg, “New quantum Poincare algebra and k deformed field theory,” *Phys.Lett.* **B293** (1992) 344–352.
  - [3] J. Lukierski and H. Ruegg, “Quantum kappa Poincare in any dimension,” *Phys.Lett.* **B329** (1994) 189–194, [arXiv:hep-th/9310117 \[hep-th\]](#).
  - [4] G. Amelino-Camelia, “Quantum-Spacetime Phenomenology,” *Living Rev.Rel.* **16** (2013) 5, [arXiv:0806.0339 \[gr-qc\]](#).
  - [5] J. Kowalski-Glikman, “De sitter space as an arena for doubly special relativity,” *Phys.Lett.* **B547** (2002) 291–296, [arXiv:hep-th/0207279 \[hep-th\]](#).
  - [6] G. Gubitosi and F. Mercati, “Relative Locality in  $\kappa$ -Poincaré,” *Class. Quant. Grav.* **30** (2013) 145002, [arXiv:1106.5710 \[gr-qc\]](#).
  - [7] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer, “Planck-scale-modified dispersion relations in homogeneous and isotropic spacetimes,” [arXiv:1612.01390 \[gr-qc\]](#).
  - [8] L. Barcaroli and G. Gubitosi, “Kinematics of particles with quantum-de Sitter-inspired symmetries,” *Phys. Rev.* **D93** (2016) no. 12, 124063, [arXiv:1512.03462 \[gr-qc\]](#).
  - [9] G. Rosati, G. Amelino-Camelia, A. Marciano, and M. Matassa, “Planck-scale-modified dispersion relations in FRW spacetime,” *Phys. Rev.* **D92** (2015) no. 12, 124042, [arXiv:1507.02056 \[hep-th\]](#).
  - [10] L. Barcaroli, L. K. Brunkhorst, G. Gubitosi, N. Loret, and C. Pfeifer, “Hamilton geometry: Phase space geometry from modified dispersion relations,” *Phys. Rev.* **D92** (2015) no. 8, 084053, [arXiv:1507.00922 \[gr-qc\]](#).
  - [11] S. Majid and H. Ruegg, “Bicrossproduct structure of kappa Poincare group and noncommutative geometry,” *Phys.Lett.* **B334** (1994) 348–354, [arXiv:hep-th/9405107 \[hep-th\]](#).
  - [12] G. Amelino-Camelia, L. Barcaroli, G. Gubitosi, S. Liberati, and N. Loret, “Realization of doubly special relativistic symmetries in Finsler geometries,” *Phys. Rev.* **D90** (2014) no. 12, 125030, [arXiv:1407.8143 \[gr-qc\]](#).
  - [13] I. P. Lobo, N. Loret, and F. Nettel, “Investigation on Finsler geometry as a generalization to curved spacetime of Planck-scale-deformed relativity in the de Sitter case,” [arXiv:1611.04995 \[gr-qc\]](#).
  - [14] M. Letizia and S. Liberati, “Deformed relativity symmetries and the local structure of spacetime,” [arXiv:1612.03065 \[gr-qc\]](#).
  - [15] U. Jacob and T. Piran, “Lorentz-violation-induced arrival delays of cosmological particles,” *JCAP* **0801** (2008) 031, [arXiv:0712.2170 \[astro-ph\]](#).

- [16] G. Amelino-Camelia, L. Barcaroli, G. D’Amico, N. Loret, and G. Rosati, “IceCube and GRB neutrinos propagating in quantum spacetime,” *Phys. Lett.* **B761** (2016) 318–325, [arXiv:1605.00496 \[gr-qc\]](#).
- [17] C. Pfeifer, “The tangent bundle exponential map and locally autoparallel coordinates for general connections on the tangent bundle with application to Finsler geometry,” *Int. J. Geom. Meth. Mod. Phys.* **13** (2016) no. 03, 1650023, [arXiv:1406.5413 \[math-ph\]](#).
- [18] E. Minguzzi, “Special coordinate systems in pseudo-Finsler geometry and the equivalence principle,” *J. Geom. Phys.* **114** (2017) 336–347, [arXiv:1601.07952 \[gr-qc\]](#).
- [19] Y. Ling, X. Li, and H.-b. Zhang, “Thermodynamics of modified black holes from gravity’s rainbow,” *Mod. Phys. Lett.* **A22** (2007) 2749–2756, [arXiv:gr-qc/0512084 \[gr-qc\]](#).
- [20] A. F. Ali, “Black hole remnant from gravity’s rainbow,” *Phys. Rev.* **D89** (2014) no. 10, 104040, [arXiv:1402.5320 \[hep-th\]](#).
- [21] C. Leiva, J. Saavedra, and J. Villanueva, “The Geodesic Structure of the Schwarzschild Black Holes in Gravity’s Rainbow,” *Mod. Phys. Lett.* **A24** (2009) 1443–1451, [arXiv:0808.2601 \[gr-qc\]](#).
- [22] Y. Gim and W. Kim, “Thermodynamic phase transition in the rainbow Schwarzschild black hole,” *JCAP* **1410** (2014) 003, [arXiv:1406.6475 \[gr-qc\]](#).